Constructive Version of Tarski’s Fixed Point Theorems
A Publication of Patrick Cousot and Radhia Cousot

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1 mathematical background (lattices, iteration sequences, ordinal numbers)
2 behaviour of iteration sequences
3 proof of the fixed point theorem
A complete Lattice \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) is

- a partially ordered set \((L, \sqsubseteq)\)
- in which every subset \(M \subset L\) has a least upper bound \(\sqcup M\)
- and a greatest lower bound \(\sqcap M\).

In this talk \(f\) should always be a monotone operator on \(L\) into itself. Remember: \(f\) monotone iff \(\forall P, Q \in L: P \sqsubseteq Q \Rightarrow f(P) \sqsubseteq f(Q)\)
Definition

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  and a greatest lower bound \(\sqcap M\).
- \(\sqcup L\) is called the supremum of \(L\) and denoted by \(\top\),
- analogously \(\sqcap L\) is called the infimum of \(L\) and denoted by \(\bot\).
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In this talk \(f\) should always be a monotone operator on \(L\) into itself.

Remember:

\(f\) monotone iff \(\forall P, Q \in L : P \sqsubseteq Q \Rightarrow f(P) \sqsubseteq f(Q)\)
Begin with an element $X^0$ of $L$
The $n$-th element of the sequence is $\underbrace{f(\ldots f(f(X^0)\ldots))}_{n \text{ times}}$
We denote this by $X^n$
Begin with an element $X^0$ of $L$

The $n$-th element of the sequence is

$$f(...f(f(X^0)...))$$

$n$–times

We denote this by $X^n$

Example:

$X^0 = u$

$f(X^0) = X^1 = v$

$f(f(X^0)) = X^2 = q$

$\langle u, v, q, y, y, y, ... \rangle$
Definition

If there is an index, starting from that all elements are the same, we call this element the limit of the sequence.

Example:

\[ X^0 = u \]
\[ f(X^0) = X^1 = v \]
\[ f(f(X^0)) = X^2 = q \]
\[ \langle u, v, q, y, y, y, \ldots \rangle \]
Some Special Points

If \( f(P) = P \) we call \( P \) **fixed point**.

If \( f(P) \sqsubseteq P \) we call \( P \) **postfixed point**.

If \( P \sqsubseteq f(P) \) we call \( P \) **prefixed point**.
Question:
Does every iteration sequence of a monotone operator has a limit?
Question:
Does every iteration sequence of a monotone operator has a limit?

Example:
⟨a, c, a, c, a, c...⟩
Question:
If a sequence has a limit in sense of calculus, does it reach its limit (in our sense of limit)?
Question:
If a sequence has a limit in sense of calculus, does it reach its limit (in our sense of limit)?

Example:

\[ L = ([-3, 3], \leq_{\mathbb{R}}, -3, 3, \text{max}_{\mathbb{R}}, \text{min}_{\mathbb{R}}) \]

\[ f(x) = \frac{x}{2} \]

\[ \langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots \rangle \]
Comparison to Limit in Sense of Calculus

Question:
If a sequence has a limit in sense of calculus, does it reach its limit (in our sense of limit)?

Example:

\[ L = ([−3, 3], \leq_{\mathbb{R}}, −3, 3, \max_{\mathbb{R}}, \min_{\mathbb{R}}) \]

\[ f(x) = \frac{x}{2} \]

\[ \langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots \rangle \]

\[ \lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0 \]
Question:
If a sequence has a limit in sense of calculus, does it reach its limit (in our sense of limit)?

Example:

\[ L = \left( [-3, 3], \leq_{\mathbb{R}}, -3, 3, \max_{\mathbb{R}}, \min_{\mathbb{R}} \right) \]
\[ f(x) = \frac{x}{2} \]
\[ \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0 \]
\[ \langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots \rangle = \langle \frac{1}{2^n} \rangle_{n \in \mathbb{N}} \]
The “Append 9” Example

\[ L = ([-3, 3], \leq_{\mathbb{R}}, -3, 3, \max_{\mathbb{R}}, \min_{\mathbb{R}}) \]

\[ f(x) = \]

- if \( x = 3 \):
  \[ f(x) = 3 \]
- else:
  Write \( x \) in decimal notation.
  Append a 9 after the last digit after the decimal point.

Conclusion:
The natural numbers are not enough to describe our iteration sequence.
The “Append 9” Example

\[ L = ([−3, 3], \leq_R, −3, 3, \max_R, \min_R) \]

\[
f(x) =
\]

- if \( x = 3 \):
  
  \( f(x) = 3 \)

- else:
  
  Write \( x \) in decimal notation.
  
  Append a 9 after the last digit after the decimal point.

\[ \langle 0, 0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots \rangle \]
The “Append 9” Example

$L = ([−3, 3], \leq_{\mathbb{R}}, −3, 3, \max_{\mathbb{R}}, \min_{\mathbb{R}})$

\[
\begin{align*}
f(x) &= \\
\text{if } x = 3: & \quad f(x) = 3 \\
\text{else:} & \\
& \text{Write } x \text{ in decimal notation.} \\
& \text{Append a 9 after the last digit after the decimal point.}
\end{align*}
\]

\[
\langle 0, 0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots \rangle
\]

Conclusion:
The natural numbers are not enough to describe our iteration sequence
What for do we use natural numbers?
What for do we use natural numbers?

- describe the size of a set
- describe the position of elements in a sequence
Beyond Natural Numbers

What for do we use natural numbers?
- describe the size of a set
- describe the position of elements in a sequence

What if a set is “too big”?

What if a sequence is “too long”?
What do we use natural numbers for?

- describe the size of a set
- describe the position of elements in a sequence

What if a set is “too big”?

- cardinal numbers

What if a sequence is “too long”?
What for do we use natural numbers?

- describe the size of a set
- describe the position of elements in a sequence

What if a set is “too big”?
~ cardinal numbers

What if a sequence is “too long”?
~ ordinal numbers
0 is a natural number.

if \( n \) is a natural number \( \text{succ}(n) \) is a natural number.
0 is a natural number.
if $n$ is a natural number $\text{succ}(n)$ is a natural number.

We want to define every natural number as a set:

- $0 := \emptyset$
- $\text{succ}(n) := n \cup \{n\}$
Construction of the Natural Numbers

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We want to define every natural number as a set:

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\begin{align*}
0 &:= \emptyset \\
1 &:= \{\emptyset\} = \{0\}
\end{align*}
\]
0 is a natural number.

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We want to define every natural number as a set:

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0 := $\emptyset$
1 := $\{\emptyset\} = \{0\}$
2 := $\{\emptyset, \{\emptyset\}\} = \{0, 1\}$
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3 & := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
\vdots
\end{align*}
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3 & := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
\vdots
\end{align*}
\]

Now we extend this by adding a new number:

\( \omega := \{0, 1, 2, 3, \ldots\} \)
- 0 is an ordinal number.
- if $n$ is an ordinal number $\text{succ}(n)$ is an ordinal number. (called successor ordinal)
Construction of the Ordinal Numbers

0 is an ordinal number.

if \( n \) is an ordinal number \( \text{succ}(n) \) is an ordinal number. (called successor ordinal)

if \( M = \{0, \ldots\} \) is closed under \( \text{succ}(n) \)
\[
\xi = \bigcup_{\alpha \in M} \alpha
\]
is an ordinal number. (called limit ordinal)
Construction of the Ordinal Numbers

- 0 is an ordinal number.
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  \[ \xi = \bigcup_{\alpha \in M} \alpha \] is an ordinal number. (called limit ordinal)

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& \vdots
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\omega := \{0, 1, 2, 3, ...\}
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\[ 2 := \{\emptyset, \{\emptyset\}\} = \{0, 1\} \]
\[ \vdots \]
\[ \omega := \{0, 1, 2, 3, \ldots\} \]
\[ \omega + 1 := \{0, 1, 2, 3, \ldots, \omega\} \]
Construction of the Ordinal Numbers

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\omega & := \{0, 1, 2, 3, \ldots\} \\
\omega + 1 & := \{0, 1, 2, 3, \ldots, \omega\} \\
\omega + 2 & := \{0, 1, 2, 3, \ldots, \omega, \omega + 1\} \\
\vdots
\end{align*}
\]
Construction of the Ordinal Numbers

- 0 is an ordinal number.
- If \( n \) is an ordinal number, \( \text{succ}(n) \) is an ordinal number. (called successor ordinal)
- If \( M = \{0, \ldots\} \) is closed under \( \text{succ}(n) \)
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\begin{align*}
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&\vdots \\
\omega &:= \{0, 1, 2, 3, \ldots\} \\
\omega + 1 &:= \{0, 1, 2, 3, \ldots, \omega\} \\
\omega + 2 &:= \{0, 1, 2, 3, \ldots, \omega, \omega + 1\} \\
&\vdots \\
\omega \cdot 2 &:= \{0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}
\end{align*}
\]
Formal Definition of Ordinal Numbers

Definition

A set $\alpha$ is an ordinal number iff

- $\alpha$ is transitive
- $(\alpha, \in)$ is well ordered
**Definition**

A set $\alpha$ is an ordinal number iff

- $\alpha$ is transitive
- $(\alpha, \in)$ is well ordered

$\alpha$ is transitive iff

every element of $\alpha$ is a subset of $\alpha$
A set $\alpha$ is an ordinal number iff

- $\alpha$ is transitive
- $(\alpha, \in)$ is well ordered

$(\alpha, \in)$ is well ordered iff
$(\alpha, \in)$ is totally ordered and every subset of $\alpha$ has a least element with respect to $\in$
addition is defined inductively
\[ \alpha + \beta = \begin{cases} 
\alpha & \text{if } \beta = 0 \\
\text{succ}(\alpha + \beta_{-1}) & \text{if } \beta = \text{succ}(\beta_{-1}) \\
\bigcup_{\gamma<\beta} \alpha + \gamma & \text{if } \beta \text{ is a limit ordinal}
\end{cases} \]

consider that addition is not commutative
\[ \omega + 3 = \text{succ}(\text{succ}(\text{succ}(\omega))) \]
\[ 3 + \omega = 3 \cup 4 \cup 5 \cup \ldots = \omega \]
analogously multiplication is defined inductivly

\[
\alpha \cdot \beta = \begin{cases} 
0 & \text{if } \beta = 0 \\
\text{succ}(\alpha \cdot \beta_{-1} + \alpha) & \text{if } \beta = \text{succ}(\beta_{-1}) \\
\bigcup_{\gamma < \beta} \alpha \cdot \gamma & \text{if } \beta \text{ is a limit ordinal}
\end{cases}
\]

consider that addition is not commutative

\[\omega \cdot 3 = \omega + \omega + \omega\]

\[3 \cdot \omega = 0 \cup 3 \cup 6 \cup 9 \cup ... = \omega\]
Ord-Indexed Iteration Sequence

Upper Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $\text{luis}_f(X^0)$
The “Append 9” Example Revisited

\[ L = ([-3, 3], \leq_{\mathbb{R}}, -3, 3, \max_{\mathbb{R}}, \min_{\mathbb{R}}) \]

\[ f(x) = \]

- if \( x = 3 \):
  \[ f(x) = 3 \]
- else:
  Write \( x \) in decimal notation.
  Append a 9 after the last digit after the decimal point.

\[ \langle 0, 0.9, 0.99, \ldots \rangle \]

\[ X^\omega = ? \]
The “Append 9” Example Revisited

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$\langle 0, 0.9, 0.99, \ldots \rangle$

$X^\omega = ?$
The “Append 9” Example Revisited

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- $X^0$ the starting value
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- $X^\delta = \bigcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $luis_f(X^0)$

\[
\langle 0, 0.9, 0.99, ..., 1 \rangle
\]

$X^\omega = 1$
The “Append 9” Example Revisited

Upper Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $\text{luis}_f(X^0)$

\[
\langle 0, 0.9, 0.99, \ldots, 1, 1.9, 1.99, \ldots \rangle
\]

$X^\omega = 1$  $X^{\omega+1} = 1.9$  $X^{\omega+2} = 1.99$
The “Append 9” Example Revisited

**Upper Iteration Sequence**

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigsqcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $\text{luis}_f(X^0)$

$$\langle 0, 0.9, 0.99, \ldots, 1, 1.9, 1.99, \ldots, 2 \rangle$$

$$X^{\omega \cdot 2} = 2$$
The “Append 9” Example Revisited

Upper Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $\text{luis}_f(X^0)$

$$\langle 0, 0.9, 0.99, \ldots, 1, 1.9, 1.99, \ldots, 2, 2.9, 2.99, \ldots \rangle$$

$$X^{\omega \cdot 2} = 2 \quad X^{\omega \cdot 2+1} = 2.9 \quad X^{\omega \cdot 2+2} = 2.99$$
The “Append 9” Example Revisited

Upper Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $luis_f(X^0)$

\[
\langle 0, 0.9, 0.99, ..., 1, 1.9, 1.99, ..., 2, 2.9, 2.99, ..., 3 \rangle
\]

$X^{\omega \cdot 3} = 3$
The “Append 9” Example Revisited

\[ L = ([−3, 3], \leq_{\mathbb{R}}, −3, 3, \max_{\mathbb{R}}, \min_{\mathbb{R}}) \]

\[ f(x) = \]

- if \( x = 3 \):
  \[ f(x) = 3 \]
- else:
  Write \( x \) in decimal notation.
  Append a 9 after the last digit after the decimal point.

\[ \langle 0, 0.9, 0.99, \ldots, 1, 1.9, 1.99, \ldots, 2, 2.9, 2.99, \ldots, 3, 3, 3, \ldots \rangle \]

\[ x^{\omega \cdot 3} = 3 \quad x^{\omega \cdot 3+1} = 3 \quad x^{\omega \cdot 3+2} = 3 \]
The “Append 9” Example Revisited

Upper Iteration Sequence

- \( X^0 \) is the starting value
- \( X^\delta = f(X^{\delta-1}) \) if \( \delta \) is a successor ordinal
- \( X^\delta = \bigcup_{\alpha < \delta} X^\alpha \) if \( \delta \) is a limit ordinal

Its limit is denoted by \( luis_f(X^0) \)

\[
\langle 0, 0.9, 0.99, ..., 1, 1.9, 1.99, ..., 2, 2.9, 2.99, ..., 3, 3, 3, ..., 3, 3, 3, ... \rangle
\]

\( X^\delta = 3 \) for all \( \delta \geq \omega \cdot 3 \) \implies \( luis_f(0) = 3 \)
Upper Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcup_{\alpha < \delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $\text{lus}_f(X^0)$
Ord-Indexed Iteration Sequence

**Upper Iteration Sequence**

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcup_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $luis_f(X^0)$

**Lower Iteration Sequence**

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcap_{\alpha<\delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $llis_f(X^0)$
Calculus Example Revisited

Lower Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta - 1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcap_{\alpha < \delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $\text{llis}_f(X^0)$

\[ L = ([-3, 3], \leq_R, -3, 3, \max_R, \min_R) \quad f(x) = x/2 \]

\[ \langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots \rangle \]

$X^\omega = ?$
Calculus Example Revisited

Lower Iteration Sequence

- \( X^0 \) the starting value
- \( X^\delta = f(X^{\delta-1}) \) if \( \delta \) is a successor ordinal
- \( X^\delta = \prod_{\alpha < \delta} X^\alpha \) if \( \delta \) is a limit ordinal

Its limit is denoted by \( llis_f(X^0) \)

\[
L =([-3, 3], \leq_R, -3, 3, max_R, min_R) \quad f(x) = x/2
\]

\[
\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots \rangle
\]

\[
X^\omega = \prod_{\delta < \omega} (\frac{1}{2})^\delta
\]
Calculus Example Revisited

Lower Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta-1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcap_{\alpha < \delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $llis_f(X^0)$

$L = ([-3, 3], \leq_R, -3, 3, \max_R, \min_R)$

$f(x) = x/2$

$\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots, 0 \rangle$

$X^\omega = \bigcap_{\delta < \omega} (\frac{1}{2})^\delta = 0$
Lower Iteration Sequence

- $X^0$ the starting value
- $X^\delta = f(X^{\delta - 1})$ if $\delta$ is a successor ordinal
- $X^\delta = \bigcap_{\alpha < \delta} X^\alpha$ if $\delta$ is a limit ordinal

Its limit is denoted by $llis_f(X^0)$

$$L = (\left[-3, 3\right], \leq_R, -3, 3, \max_R, \min_R) \quad f(x) = x/2$$

$$\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots, 0, 0, 0, \ldots, 0, 0, 0, \ldots, 0 \rangle$$

$llis_f(1) = 0$
Transfinite Induction

To prove a proposition for all natural numbers $\delta$ we use induction.

What do we have to show?

1. case $\delta = 0$:
   Proposition holds for 0 (induction basis)

2. case $\delta$ is not 0:
   Assume the proposition holds for the direct predecessor $\delta - 1$.
   Show that it also holds for $\delta$ (induction step)
Transfinite Induction

To prove a proposition for all ordinal numbers $\delta$ we use transfinite induction

What do we have to show?

1. case $\delta = 0$:
   Proposition holds for 0

2. case $\delta$ is a successor ordinal:
   Assume the proposition holds for the direct predecessor $\delta - 1$.
   Show that it also holds for $\delta$
To prove a proposition for all ordinal numbers $\delta$ we use

**transfinite induction**

What do we have to show?

1. case $\delta = 0$:
   Proposition holds for 0

2. case $\delta$ is a successor ordinal:
   Assume the proposition holds for the direct predecessor $\delta - 1$.
   Show that it also holds for $\delta$

3. case $\delta$ is a limit ordinal:
   Assume the proposition holds for all predecessors $\alpha < \delta$.
   Show that it holds also for $\delta$
1. mathematical background
   (lattices, iteration sequences, ordinal numbers)
2. behaviour of iteration sequences
3. proof of the fixed point theorem
Lemma: For all $P \in L$ holds

\[(X^0 \subseteq P \land f(P) \subseteq P) \Rightarrow \forall \delta \in \text{Ord}: X^\delta \subseteq P\]

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$:
   Proposition holds for 0
Lemma: For all \( P \in L \) holds

\[
( X^0 \sqsubseteq P \land f(P) \sqsubseteq P ) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \sqsubseteq P
\]

proof by transfinite induction over the number of iterations \( \delta \)

1. case \( \delta = 0 \):
   - Proposition holds for 0
   - \( X^0 \sqsubseteq P \) (trivial)
Lemma: For all $P \in L$ holds

\[ (X^0 \subseteq P \land f(P) \subseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \subseteq P \]

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: $\sqrt{\mbox{}}$
Lemma: For all $P \in L$ holds

$$( X^0 \subseteq P \land f(P) \subseteq P ) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \subseteq P $$

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: √

2. case $\delta$ is a successor ordinal:
   - Assume the proposition holds for the direct predecessor $\delta - 1$. Show that it also holds for $\delta$
Lemma: For all $P \in L$ holds

\[(X^0 \subseteq P \land f(P) \subseteq P) \Rightarrow \forall \delta \in Ord : X^\delta \subseteq P\]

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: $\sqrt{\text{1}}$

2. case $\delta$ is a successor ordinal:
   Assume the proposition holds for the direct predecessor $\delta - 1$. Show that it also holds for $\delta$
   Assume $X^{\delta - 1} \subseteq P$, that implies

\[f(X^{\delta - 1}) \subseteq f(P)\]
Lemma: For all $P \in L$ holds
\[
(X^0 \sqsubseteq P \land f(P) \sqsubseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \sqsubseteq P
\]
proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: $\checkmark$

2. case $\delta$ is a successor ordinal:
   Assume the proposition holds for the direct predecessor $\delta - 1$. Show that it also holds for $\delta$
   Assume $X^{\delta - 1} \sqsubseteq P$, that implies
   \[
   X^\delta = f(X^{\delta - 1}) \sqsubseteq f(P) \sqsubseteq P
   \]
Lemma: For all $P \in L$ holds

$$(X^0 \subseteq P \land f(P) \subseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \subseteq P$$

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: $\sqrt{\ }$
2. case $\delta$ is a successor ordinal: $\sqrt{\ }$
Lemma: For all $P \in L$ holds

$$\left( X^0 \subseteq P \land f(P) \subseteq P \right) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \subseteq P$$

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: $\sqrt{\square}$

2. case $\delta$ is a successor ordinal: $\sqrt{\square}$

3. case $\delta$ is a limit ordinal:
   Assume the proposition holds for all predecessors $\alpha < \delta$.
   Show that it holds also for $\delta$
Lemma: For all $P \in L$ holds

\[ (X^0 \subseteq P \land f(P) \subseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \subseteq P \]

proof by transfinite induction over the number of iterations $\delta$

1. case $\delta = 0$: $\sqrt{\square}$
2. case $\delta$ is a successor ordinal: $\sqrt{\square}$
3. case $\delta$ is a limit ordinal:
   
   Assume the proposition holds for all predecessors $\alpha < \delta$.
   Show that it holds also for $\delta$

   Assume for all $\alpha < \delta$ holds $X^\alpha \subseteq P$, that implies

   \[ X^\delta = \bigsqcup_{\alpha<\delta} X^\alpha \subseteq P \]
Lemma: For all \( P \in L \) holds

\[
(X^0 \subseteq P \wedge f(P) \subseteq P) \Rightarrow \forall \delta \in \text{Ord}: X^\delta \subseteq P
\]

proof by transfinite induction over the number of iterations \( \delta \)

1. case \( \delta = 0 \): \( \checkmark \)
2. case \( \delta \) is a successor ordinal: \( \checkmark \)
3. case \( \delta \) is a limit ordinal: \( \checkmark \)
Example: Behaviour of an Upper Iteration Sequence

\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

$X \omega \cdot 3 = X \omega \cdot 3 + 1 = \ldots = X \omega \cdot 2 + 1 = X \omega + 2 = \ldots = X \omega \cdot 2 X \omega + 1 = X \omega + 2 = \ldots$

$X^5, X^0, X^1, X^2, X^3, X^4$
Example: Behaviour of an Upper Iteration Sequence

\[ X \omega \cdot 3 = X \omega \cdot 3 + 1 = \ldots \]

\[ X \omega \cdot 2 + 1 = X \omega + 2 = \ldots \]

\[ X \omega \cdot 2 \]

\[ X^5 \]

\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

\[
X_\omega \cdot 3 = X_\omega \cdot 3 + 1 = \ldots \\
X_\omega \cdot 2 + 1 = X_\omega + 2 = \ldots \\
X_\omega \cdot 2 = X_\omega + 1 = \ldots \\
X_0, X^1, X^2, X^3, X^4
\]
Example: Behaviour of an Upper Iteration Sequence

\[ X^{\omega+1} = X^{\omega+2} = \ldots \]
\[ X^{\omega} \]

\[ \ldots \]
\[ X^5 \]
\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

\[ X^\omega \cdot 2 = X^\omega \cdot 3 + 1 = \ldots \]

\[ X^\omega \cdot 2 + 1 = X^\omega + 2 = \ldots \]

\[ X^\omega \cdot 2 \]

\[ X^{\omega + 1} = X^{\omega + 2} = \ldots \]

\[ X^\omega \]

\[ \ldots \]

\[ X^5 \]

\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

\[ X^\omega \cdot 2 + 1 = X^\omega + 2 = \ldots \]
\[ X^\omega \cdot 2 \]

\[ X^\omega + 1 = X^\omega + 2 = \ldots \]
\[ X^\omega \]

\[ \ldots \]
\[ X^5 \]
\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

\[ X^{\omega \cdot 3} = X^{\omega \cdot 3 + 1} = ... \]
\[ X^{\omega \cdot 2 + 1} = X^{\omega + 2} = ... \]
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\[ X^{\omega + 1} = X^{\omega + 2} = ... \]
\[ X^{\omega} \]

... 
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Example: Behaviour of an Upper Iteration Sequence

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\[ X^{\omega} \]

\[ \ldots \]

\[ X^5 \]

\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

\[ X^{\omega \cdot 2} = X^{\omega + 2} = \ldots \]

\[ X^{\omega \cdot 2 + 1} = X^{\omega + 2} = \ldots \]

\[ X^{\omega} \]

\[ X^{\omega + 1} = X^{\omega + 2} = \ldots \]

\[ \ldots \]

\[ X^5 \]

\[ X^0, X^1, X^2, X^3, X^4 \]
Example: Behaviour of an Upper Iteration Sequence

\[ X_\omega \cdot 3 = X_\omega \cdot 3 + 1 = ... \]
\[ X_\omega \cdot 2 + 1 = X_\omega + 2 = ... \]
\[ X_\omega \cdot 2 \]
\[ X_\omega + 1 = X_\omega + 2 = ... \]
\[ X_\omega \]

... 

\[ X^5 \]
\[ X^0, X^1, X^2, X^3, X^4 \]
Theorem

The subsequence $\langle X^{\omega \cdot \alpha}\rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$. 

Every ordinal of the form $\omega \cdot \alpha$ is a limit ordinal (or 0). So every element of the sequence is defined as $X^{\omega \cdot \alpha} = \bigsqcup_{\beta < \omega \cdot \alpha} X^\beta$. 
Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$. 

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- Every ordinal of the form \( \omega \cdot \alpha \) is a limit ordinal (or 0)
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\[
X^{\omega \cdot \alpha} = \bigsqcup_{\beta < \omega \cdot \alpha} X^\beta
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Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

Show that the sequence is not strictly increasing.
Theorem

The subsequence \( \langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}} \) is increasing and stationary. Its limit \( X^{\omega \cdot \eta} \) is the least postfixed point of \( f \) greater than or equal to \( X^0 \).

Show that the sequence is not strictly increasing.

- Assume that \( \langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}} \) is strictly increasing.
Theorem

The subsequence \( \langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}} \) is increasing and stationary. Its limit \( X^{\omega \cdot \eta} \) is the least postfixed point of \( f \) greater than or equal to \( X^0 \).

Show that the sequence is not strictly increasing.

- Assume that \( \langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}} \) is strictly increasing.
  - Let \( \mu \) be the smallest ordinal such that
    \[
    \text{Card}(\{\alpha | \alpha < \mu\}) > \text{Card}(L)
    \]
The subsequence $\langle X^{\omega \cdot \alpha}\rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

Show that the sequence is not strictly increasing.

- Assume that $\langle X^{\omega \cdot \alpha}\rangle_{\alpha \in \text{Ord}}$ is strictly increasing.
  - Let $\mu$ be the smallest ordinal such that
    \[ \text{Card}(\{\alpha | \alpha < \mu\}) > \text{Card}(L) \]
  - So $\langle X^{\omega \cdot \alpha}\rangle_{\alpha < \mu}$ is also strictly increasing.
    \[ L \supset \{X^{\omega \cdot \alpha} | \alpha < \mu\} \sim \{\omega \cdot \alpha | \alpha < \mu\} \sim \{\alpha | \alpha < \mu\} \]
Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

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  - So $\langle X^{\omega \cdot \alpha} \rangle_{\alpha < \mu}$ is also strictly increasing.
    \[ L \supset \{X^{\omega \cdot \alpha} | \alpha < \mu\} \sim \{\omega \cdot \alpha | \alpha < \mu\} \sim \{\alpha | \alpha < \mu\} \]
- Contradiction! $\Rightarrow \langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is not strictly increasing.

$\Rightarrow \exists \eta \in \mu : X^{\omega \cdot \eta} = X^{\omega \cdot (\eta+1)}$
Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

$\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is not strictly increasing.

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The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

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$\Rightarrow \exists \eta \in \mu : X^{\omega \cdot \eta} = X^{\omega \cdot (\eta+1)}$

- Since $X^{\omega \cdot (\eta+1)} = \bigcup_{\beta < \omega \cdot (\eta+1)} X^\beta$
- especially $X^{(\omega \cdot \eta)+1} \subseteq X^{\omega \cdot (\eta+1)}$
Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

$\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is not strictly increasing.

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- Since $X^{\omega \cdot (\eta+1)} = \bigsqcup_{\beta < \omega \cdot (\eta+1)} X^\beta$
  
  especially $X^{(\omega \cdot \eta)+1} \subseteq X^{\omega \cdot (\eta+1)}$

$\Rightarrow X^{\omega \cdot \eta}$ is a postfixed point of $f$ greater than or equal to $X^0$
Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

$\Rightarrow X^{\omega \cdot \eta}$ is a postfixed point of $f$ greater than or equal to $X^0$
Theorem

The subsequence \( \{X^\omega \cdot \alpha\}_{\alpha \in \text{Ord}} \) is increasing and stationary. Its limit \( X^\omega \cdot \eta \) is the least postfixed point of \( f \) greater than or equal to \( X^0 \).

\[ \Rightarrow X^\omega \cdot \eta \text{ is a postfixed point of } f \text{ greater than or equal to } X^0 \]

Lemma: For all \( P \in L \) holds

\[ (X^0 \sqsubseteq P \land f(P) \sqsubseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \sqsubseteq P \]
Theorem

The subsequence $\langle X^{\omega \cdot \alpha} \rangle_{\alpha \in \text{Ord}}$ is increasing and stationary. Its limit $X^{\omega \cdot \eta}$ is the least postfixed point of $f$ greater than or equal to $X^0$.

$\Rightarrow X^{\omega \cdot \eta}$ is a postfixed point of $f$ greater than or equal to $X^0$

Lemma: For all $P \in L$ holds

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Lemma: For all \( P \in L \) holds

\[
( X^0 \sqsubseteq P \land f(P) \sqsubseteq P ) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \sqsubseteq P
\]

\( \Rightarrow X^{\omega \cdot \eta} \) is limit

\( \Rightarrow \) stationary
Theorem

There exists a smallest limit ordinal $\xi$ such that

$X^\xi \in \text{prefp}(f) \cup \text{postfp}(f)$
Theorem (for upper iteration sequences)

Let \( \mathcal{X}^\xi \in \text{prefp}(f) \).

The subsequence \( (\mathcal{X}^\delta)^{\xi \leq \delta} \) is increasing and stationary.

Its limit \( \text{luis}_f(\mathcal{X}^\xi) \) is the least of the fixed points of \( f \) greater than or equal to \( \mathcal{X}^0 \).
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.

The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.

Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$. 
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$. The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary. Its limit $luis_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$

Show that all elements of the sequence are prefixed points:
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.
The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.
Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

Show that all elements of the sequence are prefixed points:
Assume that this is true for all $\delta$ with $\xi \leq \delta < \beta$. 
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.

The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.

Its limit $luis_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

Show that all elements of the sequence are prefixed points:
Assume that this is true for all $\delta$ with $\xi \leq \delta < \beta$.

- case $\beta$ is successor ordinal
Theorem (for upper iteration sequences)

Let \( X^\xi \in \text{prefp}(f) \).

The subsequence \( (X^\delta)_{\xi \leq \delta} \) is increasing and stationary. Its limit \( \text{luis}_f(X^\xi) \) is the least of the fixed points of \( f \) greater than or equal to \( X^0 \)

Show that all elements of the sequence are prefixed points:
Assume that this is true for all \( \delta \) with \( \xi \leq \delta < \beta \)

- case \( \beta \) is successor ordinal
  \[
  X^{\beta-1} \sqsubseteq X^\beta \quad f \text{ monotone} \quad \Rightarrow \quad X^\beta \sqsubseteq X^{\beta+1}
  \]
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.
The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.
Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$

Show that all elements of the sequence are prefixed points:
Assume that this is true for all $\delta$ with $\xi \leq \delta < \beta$

- case $\beta$ is successor ordinal
  $X^{\beta-1} \subseteq X^\beta$ $f$ monotone $\Rightarrow$ $X^\beta \subseteq X^{\beta+1}$

- case $\beta$ is limit ordinal
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.
The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.
Its limit $luis_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

Show that all elements of the sequence are prefixed points:
Assume that this is true for all $\delta$ with $\xi \leq \delta < \beta$.

- **case $\beta$ is successor ordinal**
  \[ X^{\beta-1} \subseteq X^\beta \text{ monotone } \Rightarrow X^\beta \subseteq X^{\beta+1} \]

- **case $\beta$ is limit ordinal**
  \[ X^\beta = \bigcup_{\xi \leq \alpha < \beta} X^\alpha \]
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$. The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary. Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

Show that all elements of the sequence are prefixed points:
Assume that this is true for all $\delta$ with $\xi \leq \delta < \beta$

- case $\beta$ is successor ordinal
  \[X^{\beta-1} \subseteq X^\beta \quad f \text{ monotone} \quad X^\beta \subseteq X^{\beta+1}\]

- case $\beta$ is limit ordinal
  \[X^\beta = \bigsqcup_{\xi \leq \alpha < \beta} X^\alpha\]
  \[f \text{ monotone} \quad \forall \alpha \text{ such that } \xi \leq \alpha < \beta \text{ holds: } f(X^\alpha) \subseteq f(X^\beta)\]
Theorem (for upper iteration sequences)

Let \( X^\xi \in \text{pref}(f) \).
The subsequence \((X^\delta)_{\xi \leq \delta}\) is increasing and stationary.
Its limit \( luis_f(X^\xi)\) is the least of the fixed points of \( f \) greater than or equal to \( X^0 \).

Show that all elements of the sequence are prefixed points:
Assume that this is true for all \( \delta \) with \( \xi \leq \delta < \beta \):

1. **Case \( \beta \) is successor ordinal**
   \[
   X^{\beta - 1} \sqsubseteq X^\beta \quad f \text{ monotone} \quad X^\beta \sqsubseteq X^{\beta + 1}
   \]

2. **Case \( \beta \) is limit ordinal**
   \[
   X^\beta = \bigsqcup_{\xi \leq \alpha < \beta} X^\alpha
   \]
   \[
   f \text{ monotone} \quad \forall \alpha \text{ such that } \xi \leq \alpha < \beta \text{ holds: } f(X^\alpha) \sqsubseteq f(X^\beta)
   \]
   \[
   f(X^\alpha) \sqsubseteq X^\beta \quad \Rightarrow \quad X^\beta \sqsubseteq f(X^\beta)
   \]
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$. The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary. Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

All elements of the sequence are elements of $\text{prefp}(f)$. 
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$. The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary. Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

All elements of the sequence are elements of $\text{prefp}(f)$. The sequence is not strictly increasing (trick with $\mu$).
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.

The subsequence $(X^\delta)^{\xi \leq \delta}$ is increasing and stationary.

Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

All elements of the sequence are elements of $\text{prefp}(f)$.

The sequence is not strictly increasing (trick with $\mu$)

$\Rightarrow$ After some iterations the sequence reaches a fixed point.
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$. The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary. Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

All elements of the sequence are elements of $\text{prefp}(f)$. The sequence is not strictly increasing (trick with $\mu$) $\Rightarrow$ After some iterations the sequence reaches a fixed point.

Lemma: For all $P \in L$ holds

$$( X^0 \sqsubseteq P \quad \land \quad f(P) \sqsubseteq P ) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \sqsubseteq P $$
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.
The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.
Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

All elements of the sequence are elements of $\text{prefp}(f)$.
The sequence is not strictly increasing (trick with $\mu$).
$\Rightarrow$ After some iterations the sequence reaches a fixed point.

Lemma: For all $P \in L$ holds

$(X^0 \subseteq P \land f(P) \subseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \subseteq P$

$\Rightarrow$ The sequence is stationary
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.

The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary.

Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

Show that

$\text{luis}_f(X^\xi) \sqsubseteq P$ for every postfixed point $P$ such that $X^0 \sqsubseteq P$. 
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$. The subsequence $(X^\delta)_{\xi \leq \delta}$ is increasing and stationary. Its limit $luis_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

Show that $luis_f(X^\xi) \sqsubseteq P$ for every postfixed point $P$ such that $X^0 \sqsubseteq P$.

Lemma: For all $P \in L$ holds

$$(X^0 \sqsubseteq P \land f(P) \sqsubseteq P) \Rightarrow \forall \delta \in \text{Ord} : X^\delta \sqsubseteq P$$
Theorem (for upper iteration sequences)

Let $X^\xi \in \text{prefp}(f)$.
The subsequence $(X^\delta)^{\xi \leq \delta}$ is increasing and stationary.
Its limit $\text{luis}_f(X^\xi)$ is the least of the fixed points of $f$ greater than or equal to $X^0$.

By duality we get:

Theorem (for lower iteration sequences)

Let $X^\xi \in \text{postfp}(f)$.
The subsequence $(X^\delta)^{\xi \leq \delta}$ is decreasing and stationary.
Its limit $\text{llis}_f(X^\xi)$ is the greatest of the fixed points of $f$ less than or equal to $X^0$. 
Upper Closure Operator

An operator $\bar{\rho}$ on $L$ into itself is called an upper closure operator $\bar{\rho}$ iff

- $\bar{\rho}$ is monotone
- $\bar{\rho}$ is extensive i.e. $\forall P \in L: P \subseteq \bar{\rho}(P)$
- $\bar{\rho}$ is idempotent i.e. $\forall P \in L : \bar{\rho}(\bar{\rho}(P)) = \bar{\rho}(P)$
Corollary

The restriction of $luis_f$ to $prefp(f)$ is an upper closure operator.
Corollary
The restriction of \( luis_f \) to \( prefp(f) \) is an upper closure operator.

Upper Closure Operator
An operator \( \bar{\rho} \) on \( L \) into itself is called an upper closure operator \( \bar{\rho} \) iff

- \( \bar{\rho} \) is monotone
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An operator $\rho$ on $L$ into itself is called a lower closure operator $\rho$ iff

- $\rho$ is monotone
- $\rho$ is reductive i.e. $\forall P \in L : \rho(P) \subseteq P$
- $\rho$ is idempotent i.e. $\forall P \in L : \rho(\rho(P)) = \rho(P)$
An operator $\rho$ on $L$ into itself is called an lower closure operator $\rho$ iff

- $\rho$ is monotone
- $\rho$ is reductive i.e. $\forall P \in L : \rho(P) \sqsubseteq P$
- $\rho$ is idempotent i.e. $\forall P \in L : \rho(\rho(P)) = \rho(P)$

By duality principle we get:

**Corollary**

The restriction of $llis_f$ to $postfp(f)$ is a lower closure operator.
A New Operator

\[ \lambda X. X \sqcup f(X) \]
A New Operator

\[ \lambda X. X \sqcup f(X) \]
Theorem

\[ luis_{\lambda X. X \sqcup f(X)}(L) = \text{postfp}(f) \]
Theorem

\[ \text{luis}_{\lambda X. X \sqcup f(X)}(L) = \text{postfp}(f) \]

\(P\) is fixed point of \(\lambda X. X \sqcup f(X)\) \iff \(P\) is postfixed point of \(f\)
Theorem

\[ luis_{\lambda X. X \sqcup f(X)}(L) = \text{postfp}(f) \]

\[ P \text{ is fixed point of } \lambda X. X \sqcup f(X) \iff P \text{ is postfixed point of } f \]

(dual) Theorem

\[ llis_{\lambda X. X \sqcap f(X)}(L) = \text{prefp}(f) \]
Theorem

\( luis_{\lambda X. X \sqcup f(X)} \) is an upper closure operator.
Theorem

\( luis_{\lambda X. X \sqcup f(X)} \) is an upper closure operator.

(dual) Theorem

\( llis_{\lambda X. X \sqcap f(X)} \) is a lower closure operator.
1. mathematical background
   (lattices, iteration sequences, ordinal numbers)
2. behaviour of iteration sequences
3. proof of the fixed point theorem
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)\]
Let $(L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ be a complete lattice.

**Theorem**

The image of $L$ under an upper closure operator $\bar{\rho}$ is the complete lattice:

$$(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)$$

- Show that any subset $M \subset \bar{\rho}(L)$ has the l.u.b. $\bar{\rho}(\sqcup M)$
- Show that for any subset $M \subset \bar{\rho}(L)$ holds $\bar{\rho}(\sqcap M) = \sqcap M$
Let \((L, \sqsubset, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[
(\bar{\rho}(L), \sqsubset, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)
\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\)
  - Let \(\bar{\rho}(y)\) be any upper bound of \(M\).
  
  \[
  \Rightarrow \quad \sqcup M \leq \bar{\rho}(y)
  \]
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\)

  Let \(\bar{\rho}(y)\) be any upper bound of \(M\).

  \[
  \Rightarrow \quad \sqcup M \leq \bar{\rho}(y) \\
  \bar{\rho} \text{ monotone} \quad \Rightarrow \quad \bar{\rho}(\sqcup M) \leq \bar{\rho}(\bar{\rho}(y))
  \]
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[ (\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap) \]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\)
  
  Let \(\bar{\rho}(y)\) be any upper bound of \(M\).

  \[ \Rightarrow \quad \sqcup M \leq \bar{\rho}(y) \]

  \(\bar{\rho}\) monotone \(\Rightarrow\)

  \[ \bar{\rho}(\sqcup M) \leq \bar{\rho}(\bar{\rho}(y)) = \bar{\rho}(y) \]
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\) \(\checkmark\)
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[
(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)
\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\)
- Show that for any subset \(M \subset \bar{\rho}(L)\) holds \(\bar{\rho}(\sqcap M) = \sqcap M\)
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M. \bar{\rho}(\sqcup M), \sqcap)\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\) \(\checkmark\)
- Show that for any subset \(M \subset \bar{\rho}(L)\) holds \(\bar{\rho}(\sqcap M) = \sqcap M\)

\[\forall x \in M : \quad \sqcap M \sqsubseteq x\]
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\) \(\checkmark\)
- Show that for any subset \(M \subset \bar{\rho}(L)\) holds \(\bar{\rho}(\sqcap M) = \sqcap M\)

\[\forall x \in M : \quad \sqcap M \sqsubseteq x\]

\[\bar{\rho} \text{ monotone} \quad \Rightarrow \quad \forall x \in M : \quad \bar{\rho}(\sqcap M) \sqsubseteq \bar{\rho}(x)\]
Let \((L, \subseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[
(\bar{\rho}(L), \subseteq, \bar{\rho}(\bot), \top, \lambda M. \bar{\rho}(\sqcup M), \sqcap)
\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\) \(\sqrt{\quad}\)
- Show that for any subset \(M \subset \bar{\rho}(L)\) holds \(\bar{\rho}(\sqcap M) = \sqcap M\)

\[
\begin{align*}
\bar{\rho} \text{ monotone} & \quad \forall x \in M : \quad \sqcap M \subseteq x \\
\bar{\rho} \text{ idempotent} & \quad \forall x \in M : \quad \bar{\rho}(\sqcap M) \subseteq \bar{\rho}(x)
\end{align*}
\]
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[
(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M. \bar{\rho}(\sqcup M), \sqcap)
\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\)  
- Show that for any subset \(M \subset \bar{\rho}(L)\) holds \(\bar{\rho}(\sqcap M) = \sqcap M\)

\[
\forall x \in M : \quad \sqcap M \sqsubseteq x
\]

- \(\bar{\rho}\) monotone  
  \[
  \Rightarrow \quad \forall x \in M : \quad \bar{\rho}(\sqcap M) \sqsubseteq \bar{\rho}(x)
  \]

- \(\bar{\rho}\) idempotent  
  \[
  \Rightarrow \quad \forall x \in M : \quad \bar{\rho}(\sqcap M) \sqsubseteq x
  \]

- \(\bar{\rho}\) extensive  
  \[
  \Rightarrow \quad \forall x \in M : \quad \sqcap M \sqsubseteq \bar{\rho}(\sqcap M) \sqsubseteq x
  \]
Let \((L, \sqsubseteq, \perp, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\perp), \top, \lambda M. \bar{\rho}(\sqcup M), \sqcap)\]

- Show that any subset \(M \subset \bar{\rho}(L)\) has the l.u.b. \(\bar{\rho}(\sqcup M)\) \(\checkmark\)
- Show that for any subset \(M \subset \bar{\rho}(L)\) holds \(\bar{\rho}(\sqcap M) = \sqcap M\)

\(\bar{\rho}\) monotone \(\Rightarrow\)
\[\forall x \in M : \quad \sqcap M \sqsubseteq x\]

\(\bar{\rho}\) idempotent \(\Rightarrow\)
\[\forall x \in M : \quad \bar{\rho}(\sqcap M) \sqsubseteq \bar{\rho}(x)\]

\(\bar{\rho}\) extensive \(\Rightarrow\)
\[\forall x \in M : \quad \sqcap M \sqsubseteq \bar{\rho}(\sqcap M) \sqsubseteq x\]

\(\Rightarrow \quad \bar{\rho}(\sqcap M) = \sqcap M\)
Let \((L, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) be a complete lattice.

**Theorem**

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[
(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\bot), \top, \lambda M.\bar{\rho}(\sqcup M), \sqcap)
\]

By duality we get:

**Theorem**

The image of \(L\) under an lower closure operator \(\rho\) is the complete lattice:

\[
(\bar{\rho}(L), \sqsubseteq, \bot, \bar{\rho}(\top), \sqcup, \lambda M.\bar{\rho}(\sqcap M))
\]
The image of \( \llis_{\lambda x. d \sqcap f(x)}(L) \) is the complete lattice:

\[
(prefp(f), \sqsubseteq, \bot, ?, \sqcup, ?)
\]
Theorem

The image of $\text{llis}_{\lambda x. d \sqcap f(x)}(L)$ is the complete lattice:

$$(\text{prefp}(f), \subseteq, \bot, ?, \sqcup, ?)$$

(dual) Theorem

$\text{llis}_{\lambda X. X \sqcap f(X)}(L) = \text{prefp}(f)$

(dual) Theorem

$\text{llis}_{\lambda X. X \sqcap f(X)}$ is an lower closure operator.

Theorem

The image of $L$ under an lower closure operator $\rho$ is the complete lattice:

$$(\bar{\rho}(L), \subseteq, \bot, \bar{\rho}(\top), \sqcup, \lambda M. \bar{\rho}(\sqcap M))$$
(nearly) The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M.luis_f(\sqcup M), ?)\) is a complete lattice
The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M.\ luis_f(\sqcup M), ?)\) is a complete lattice

Theorem

*The image of \(llis_{\lambda x.\cdot\sqcap f(x)}(L)\) is the complete lattice:*

\((prefp(f), \sqsubseteq, \bot, ?, \sqcup, ?)\)
(nearly) The fixed point theorem

\[(fp(f), \sqsubseteq, luis_f(\perp), ?, \lambda M. luis_f(\sqcup M), ?)\] is a complete lattice

We know:

\[(prefp(f), \sqsubseteq, \perp, ?, \sqcup, ?)\] is a complete lattice.
The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M. luis_f(\sqcup M), ?)\) is a complete lattice

We know:

\((\text{prefp}(f), \sqsubseteq, \bot, ?, \sqcup, ?)\) is a complete lattice.

Theorem (for upper iteration sequences)

Let \(X^\xi \in \text{prefp}(f)\).

The subsequence \((X^\delta)_{\xi \leq \delta}\) is increasing and stationary.

Its limit \(luis_f(X^\xi)\) is the least of the fixed points of \(f\) greater than or equal to \(X^0\)

We know:

\(luis_f(\text{prefp}(f)) = fp(f)\)
(nearly) The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M.luis_f(\sqcup M), ?)\) is a complete lattice

We know:

\((prefp(f), \sqsubseteq, \bot, ?, \sqcup, ?)\) is a complete lattice.

We know:

\(luis_f(prefp(f)) = fp(f)\)
The fixed point theorem

\[(fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M. luis_f(\sqcup M), ?)\] is a complete lattice

We know:

\[(prefp(f), \sqsubseteq, \bot, ?, \sqcup, ?)\] is a complete lattice.

We know:

\[luis_f(prefp(f)) = fp(f)\]

Corollary

The restriction of \(luis_f\) to \(prefp(f)\) is an upper closure operator.
(nearly) The fixed point theorem

\[(fp(f), \sqsubseteq, luis_f(\perp), ?, \lambda M. luis_f(\sqcup M), ?) \text{ is a complete lattice}\]

We know:

\[(prefp(f), \sqsubseteq, \perp, ?, \sqcup, ?) \text{ is a complete lattice.}\]

We know:

\[luis_f(prefp(f)) = fp(f)\]

Corollary

The restriction of \(luis_f\) to \(prefp(f)\) is an upper closure operator.

Theorem

The image of \(L\) under an upper closure operator \(\bar{\rho}\) is the complete lattice:

\[(\bar{\rho}(L), \sqsubseteq, \bar{\rho}(\perp), \top, \lambda M. \bar{\rho}(\sqcup M), \sqcap)\]
(nearly) The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M. luis_f(\sqcup M), ?)\) is a complete lattice

By duality we get:

(nearly) The fixed point theorem (dual version)

\((fp(f), \sqsubseteq, ?, llis_f(\top), ?, \lambda M. llis_f(\sqcap M))\) is a complete lattice
The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), llis_f(\top), \lambda M. luis_f(\sqcup M), \lambda M. llis_f(\sqcap M))\)

is a complete lattice

(nearly) The fixed point theorem

\((fp(f), \sqsubseteq, luis_f(\bot), ?, \lambda M. luis_f(\sqcup M), ?)\) is a complete lattice

(nearly) The fixed point theorem (dual version)

\((fp(f), \sqsubseteq, ?, llis_f(\top), ?, \lambda M. llis_f(\sqcap M))\) is a complete lattice
used ordinal numerers to describe iteration sequences
proofed that the image of an upper closure operator is a complete lattice
found an lower closure operator that mapped L to pref(f)
found an upper closure operator that mapped pref(f) to fp(f)